Assignment

Q1 $S_{t_0}=100 \quad u=1.2 \quad d=0.9 \quad 1+r\Delta t=1.1, \quad K=110$

American put

$$\begin{tikzcd}[column sep=small]
 & & S_{t_0}u^2 \ (144) \quad f_{uu} & 0 \\
 & S_{t_0}u \ (120) \quad f_u, \phi_u \arrow[ur] \arrow[dr] & & \\
S_{t_0} \ (100) \quad f, \phi \arrow[ur] \arrow[dr] & & S_{t_0}ud / S_{t_0}du \ (108) \quad f_{ud}/f_{du} & 2 \\
 & S_{t_0}d \ (90) \quad f_d, \phi_d \arrow[ur] \arrow[dr] & & \\
 & & S_{t_0}d^2 \ (81) \quad f_{dd} & 29
\end{tikzcd}$$

Expected payoff: 0, 2, 29

$$q=\frac{1.1-0.9}{1.2-0.9}\approx 0.67$$

$$f_u = \max\left((K-S_{t_0}u)_+,\ (1+r\Delta t)^{-1}(f_{uu}q + f_{ud}\times(1-q))\right)$$
$$=\max(0,\ 1.1^{-1}(0\times0.67+2\times0.33))$$
$$=0.6$$

$$\phi_u=\frac{0-2}{144-108}\approx -0.05$$ $\Rightarrow \phi_u = 0$ since option has been exercised at time t_0

$$f_d=\max\left((K-S_{t_0}d)_+,\ (1+r\Delta t)^{-1}(f_{du}q+f_{dd}(1-q))\right)$$
$$=\max(20,\ 1.1^{-1}(2\times0.67+29\times0.33))$$
$$=\max(20,\ 9.92)$$
$$=20$$

$$\phi_d=0$$

$$f=\max\left((K-S_{t_0})_+,\ (1+r\Delta t)^{-1}(f_u q+f_d(1-q))\right)$$
$$=\max(10,\ 1.1^{-1}(0.6\times0.67+20\times0.33))$$
$$=\max(10,\ 6.37)$$
$$=10$$

$\phi=0$ because it exercised at time t_0

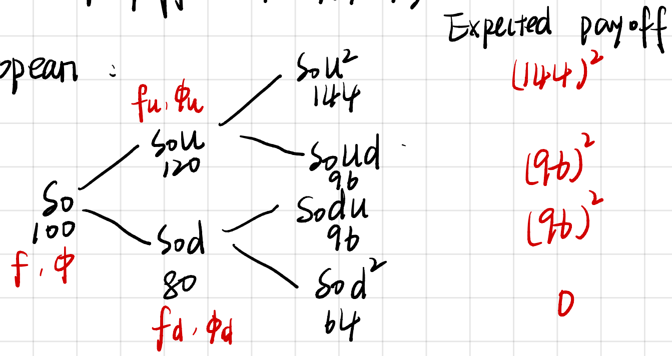
Therefore, price is 10 at time t_0

Strategy: American put option has been exercised at time t_0.

Q2 $r=0.05$ $S_0=100$, $u=1.2$ $d=0.8$ $\Delta t=1$

payoff $S_T^2 1_{\{S_T>90\}}$

(a) European :



$$q=\frac{(1+r\Delta t)-d}{u-d}=\frac{1.05-0.8}{1.2-0.8}=0.625$$

$$f_u=(1.05)^{-1}((144)^2\times 0.625+(96)^2\times 0.375)$$

$$\approx 15634.29$$

$$\phi_u=\frac{(144)^2-(96)^2}{144-96}=240$$

$$f_d=(1.05)^{-1}((96)^2\times 0.625+0\times 0.375)$$

$$\approx 5485.71$$

$$\phi_d=\frac{(96)^2-0}{96-64}=288$$

$$f=(1.05)^{-1}(15634.29\times 0.625+5485.71\times 0.375)$$

$$\approx 11265.30$$

$$\phi=\frac{15634.29-5485.71}{120-80}=253.7145$$

Therefore, for the European call option, the price is 11265.30

Strategy : at time t_0 : borrow 14106.15 from bank and buy 253.7145 shares of stock.

at time t_1, if stock goes up, sell 13.7145 shares of stock and deposit all cash

if stock goes down, buy 34.2855 shares of stock and borrow 2742.84 from bank

(b) for American call option.

$$f_u = \max\left((S_0u)^2 \cdot 1_{\{S_0u > 90\}} , 15634.29 \right)$$

$$= \max((120)^2, 15634.29)$$

$$= 15634.29$$ not exercise at this node.

$$\phi_u = \frac{(144)^2 - (96)^2}{144 - 96} = 240$$

$$f_d = \max\left((S_0d)^2 1_{\{S_0d > 90\}} , 5485.71 \right)$$

$$= \max(0, 5485.71)$$

$$= 5485.71$$

not exercise at this node.

$$\phi_d = \frac{(96)^2 - 0}{96 - 64} = 288$$

$$f = \max\{ S_0^2 1_{\{S_0 > 90\}} , 11265.3)$$

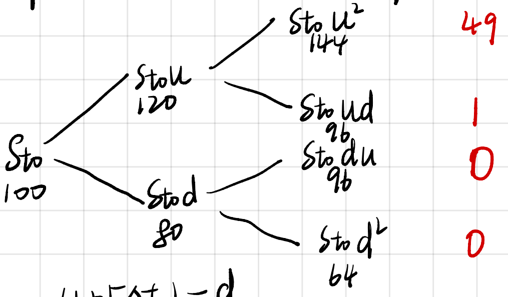
$$= 11265.3$$

$$\phi = \frac{15634.29 - 5485.71}{120 - 80} = 253.7145$$

Because American option will not exercise before $t=2$, the price is same as European option. Strategy is also same

Q3 $S_{t_0} = 100 \quad u = 1.2 \quad d = 0.8 \quad r = 0.05 \quad \Delta t = 1$

European knock-out call option $\quad K = 95,\ H = 90$, payoff $= (S_{t_2} - K)_+ \mathbb{1}\{\min_{k \in \{0,1,2\}} S_{t_k} > H\}$



$$q = \frac{(1 + r\Delta t) - d}{u - d} = \frac{1.05 - 0.8}{1.2 - 0.8} = 0.625$$

$$f_u = (1.05)^{-1}(49 \times 0.625 + 1 \times 0.375) \approx 29.52$$

$$\phi_u = \frac{49 - 1}{144 - 96} = 1$$

$$f_d = (1.05)^{-1}(0 \times 0.625 + 0 \times 0.375) = 0.$$

$$\phi_d = 0$$

$$f = (1.05)^{-1}(29.52 \times 0.625 + 0 \times 0.375) \approx 17.57$$

$$\phi = \frac{29.52 - 0}{120 - 80} = 0.738$$

The price should be 17.57 at time t_0

Strategy: At time t_0, you should borrow 56.23 from bank and long 0.738 shares of stock.

At time t_1, if stock price goes up, you should borrow 31.44 from bank and long extra 0.262 shares of stock

if stock price goes down, you should sell all of stocks and put all cash into bank.

Q4 $X \sim N(\mu, \sigma^2)$

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

set $z = \frac{x-\mu}{\sigma}$, so that $x = \sigma z + \mu$ $\quad dx = \sigma \, dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu) - \frac{1}{2} z^2} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{t\sigma z - \frac{1}{2} z^2} \, dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\sigma t)}{2}} \, dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$